

# POSITION VECTORS OF NUMERICAL SEMIGROUPS

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ABSTRACT. We provide a new way to represent numerical semigroups by showing that the position of every Apéry set of a numerical semigroup  $S$  in the enumeration of the elements of  $S$  is unique, and that  $S$  can be re-constructed from this “position vector.” We extend the discussion to more general objects called numerical sets, and show that there is a one-to-one correspondence between  $m$ -tuples of positive integers and the position vectors of numerical sets closed under addition by  $m + 1$ . We consider the problem of determining which position vectors correspond to numerical semigroups.

## 1. INTRODUCTION

We let  $\mathbb{N}$  and  $\mathbb{N}_0$  denote the positive and nonnegative integers, respectively. A *numerical semigroup*  $S$  is a subsemigroup of  $\mathbb{N}_0$  that contains 0 and has finite complement in  $\mathbb{N}_0$ . For two elements  $u$  and  $u'$  in  $S$ ,  $u \preceq_S u'$  if there exists an  $s \in S$  such that  $u + s = u'$ . This defines a partial ordering on  $S$ . The minimal elements in  $S \setminus \{0\}$  with respect to this ordering form a unique minimal set of generators for  $S$ , which is denoted by  $\{a_1, a_2, \dots, a_\nu\}$  where  $a_1 < a_2 < \dots < a_\nu$ . The semigroup  $S = \{\sum_{i=1}^\nu c_i a_i : c_i \geq 0\}$  is represented using the notation  $S = \langle a_1, \dots, a_\nu \rangle$ . Since the minimal generators of  $S$  are distinct modulo  $a_1$ , the set of minimal generators is finite. Furthermore, having finite complement in  $\mathbb{N}_0$  is equivalent to  $\gcd\{a_i : 1 \leq i \leq \nu\} = 1$ .

The number of minimal generators of a semigroup  $S$  is called the *embedding dimension* of  $S$ , and is denoted by  $\nu = \nu(S)$ . The element  $a_1$  is called the *multiplicity* of  $S$ , and is also denoted by  $e_0(S)$ . When  $S \neq \mathbb{N}_0$ , we always have  $2 \leq \nu(S) \leq e_0(S)$ .

For  $0 \neq n \in S$ , the *Apéry set* of  $S$  with respect to  $n$  is the set

$$\text{Ap}(S, n) = \{w \in S : w - n \notin S\}.$$

Every numerical semigroup containing  $n$  has a unique Apéry set with respect to  $n$  from which much can be gleaned. Indeed, in [3] the Apéry set is described as the most versatile tool in numerical semigroup theory.

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We take the representation by an Apéry set a step further. Consider the semigroup  $S = \langle 4, 7, 9 \rangle$ . We have  $\text{Ap}(S, 4) = \{0, 7, 9, 14\}$ . If we enumerate the elements of  $S$  so that  $S = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ , where  $\lambda_i < \lambda_j$  whenever  $i < j$ , then  $\text{Ap}(S, 4) = \{\lambda_0, \lambda_2, \lambda_4, \lambda_8\}$ . We can say that the position of the Apéry set in the enumeration is given by  $(0, 2, 4, 8)$ . It will be convenient to remove 0 from this vector and consider the difference of the components. For example, we represent  $S = \langle 4, 7, 9 \rangle$  (as a semigroup containing 4) with the vector  $(2, 2, 4)$  instead of  $(0, 2, 4, 8)$ . This new vector has nonnegative integer components and stores equivalent information about the semigroup. We make the following definition.

**Definition 1.1.** Let  $S = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$  be a numerical semigroup containing  $n \neq 0$  such that  $\lambda_i < \lambda_j$  whenever  $i < j$ . If  $\text{Ap}(S, n) = \{\lambda_0, \lambda_{x_1}, \lambda_{x_2}, \dots, \lambda_{x_{n-1}}\}$ , then the  $(n-1)$ -tuple  $(x_1, x_2 - x_1, x_3 - x_2, \dots, x_{n-1} - x_{n-2})$  is called the *position vector* of  $S$  with respect to  $n$ , and is denoted by  $\text{pv}_n(S)$ .

We show in Corollary 2.11 that if  $S$  and  $T$  are semigroups containing  $n \neq 0$ , then  $S = T$  if and only if  $\text{pv}_n(S) = \text{pv}_n(T)$ . Thus, a position vector is a representation of the semigroup. This is a rather remarkable fact. Consider the semigroup  $\langle 4, 7, 9 \rangle$ . Since the position vector is  $(2, 2, 4)$  (which means that the elements of  $\text{Ap}(\langle 4, 7, 9 \rangle, 4)$  are in positions 0, 2, 4, and 8 in the enumeration), no other semigroup can have an Apéry set with elements in those same positions. This is certainly not true for minimal generating sets: the position of the minimal generators of  $\langle 4, 7, 9 \rangle$  and  $\langle 4, 6, 9 \rangle$  are the same, namely the first, second, and fourth elements in the enumeration. Nonetheless, the position vectors are  $(2, 2, 4)$  and  $(2, 2, 5)$  respectively.

Not every vector of positive integers is the position vector of a numerical semigroup. In Section 2 we extend our consideration to more general objects than numerical semigroups, which we call numerical sets, and show that there is a one-to-one correspondence between elements of  $\mathbb{N}^{n-1}$  and the position vectors of numerical sets closed under addition by  $n$ . Although we do not stress the fact in this paper, a numerical set  $I$  can always be interpreted as a relative ideal of a semigroup contained in  $I$ , and so, in a sense, we have not extended beyond the theory of numerical semigroups.

Among the position vectors of numerical sets, we examine the problem of determining which represent numerical semigroups in Section 3.

## 2. POSITION VECTORS OF NUMERICAL SETS

As stated in the introduction, we need to work with objects more general than numerical semigroups.

**Definition 2.1.** A *numerical set*  $I$  is a subset of  $\mathbb{N}_0$  that contains 0 and has finite complement in  $\mathbb{N}_0$ .

**Remark 2.2.** A numerical set  $I$  is closed under addition if and only if it is a numerical semigroup. When  $I$  is not closed under addition, it is a relative ideal of the numerical semigroup  $I - I$ . Thus, we can think of the numerical sets as a certain collection of relative ideals which includes numerical semigroups. See [1] for information on relative ideals and [5, 6] for numerical sets.

We need to define the position vector of a numerical set as we did for numerical semigroups in the introduction. We begin with a preliminary definition.

**Definition 2.3.** For  $n \neq 0$ , let  $\Gamma_n$  be the collection of numerical sets  $I$  such that  $n + i \in I$  for all  $i \in I$ .

For  $I \in \Gamma_n$ , we can define the Apéry set with respect to  $n$  as

$$\text{Ap}(I, n) = \{w \in I : w - n \notin I\}.$$

As with numerical semigroups, it is not difficult to see that  $\text{Ap}(I, n)$  contains exactly  $n$  elements of  $I$  including 0. Now we can define the position vector of the numerical set.

**Definition 2.4.** Let  $I = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$  be in  $\Gamma_n$  such that  $\lambda_i < \lambda_j$  whenever  $i < j$ . If  $\text{Ap}(I, n) = \{\lambda_0, \lambda_{x_1}, \lambda_{x_2}, \dots, \lambda_{x_{n-1}}\}$ , the  $(n-1)$ -tuple  $(x_1, x_2 - x_1, x_3 - x_2, \dots, x_{n-1} - x_{n-2})$  is called the *position vector* of  $I$  with respect to  $n$ , and is denoted by  $\text{pv}_n(I)$ .

A numerical set has multiple position vectors, but no two can have the same length. Therefore,  $f : \Gamma_n \mapsto \mathbb{N}^{n-1}$  such that  $f(I) = \text{pv}_n(I)$  is a well-defined function. The goal of this section is to prove that  $f$  is a one-to-one correspondence, which is proven in Theorem 2.10.

We first establish Proposition 2.6, which contains a result about permutations on the set  $1, 2, \dots, m$ .

**Definition 2.5.** Let  $\pi = [\pi_1 \cdots \pi_m]$  be a permutation of the set  $\{1, \dots, m\}$ . Then the *conversion vector* of  $\pi$  is  $r = (r_1, \dots, r_m)$ , where  $r_i = |\{j : j < i \text{ and } \pi_j < \pi_i\}|$ .

Notice that in Definition 2.5, we have  $0 \leq r_i \leq i - 1$  for all  $1 \leq i \leq m$ . Moreover, Proposition 2.6 reveals that every vector  $(r_1, \dots, r_m)$  with this restriction is the conversion vector of a unique permutation. This result is similar to a well-known result about inversion vectors of permutations, see [7, 8].

**Proposition 2.6.** For a fixed integer  $m \geq 1$ , let  $r = (r_1, r_2, \dots, r_m)$  be a vector such that  $0 \leq r_i \leq i - 1$ , for  $1 \leq i \leq m$ . Then there is a unique permutation  $\pi$  of the set  $\{1, \dots, m\}$  for which  $r$  is its conversion vector.

*Proof.* Since there are exactly  $m!$  such vectors and  $m!$  permutation of length  $m$ , it suffices to show that each vector is the conversion vector of some permutation. The uniqueness follows by counting.

We will proceed by induction on  $m$ . If  $m = 1$ , then  $r = (0)$  and  $\pi = [1]$  has  $r$  as its conversion vector. Now assume that  $m \geq 2$  and that there is a permutation  $\sigma = [\sigma_1 \sigma_2 \cdots \sigma_{m-1}]$  with the conversion vector  $(r_1, r_2, \dots, r_{m-1})$ . For each  $1 \leq i \leq m - 1$ , define

$$\delta_i = \begin{cases} 1 & \text{if } \sigma_i > r_m \\ 0 & \text{otherwise.} \end{cases}$$

Let  $\pi = [(\sigma_1 + \delta_1)(\sigma_2 + \delta_2) \cdots (\sigma_{m-1} + \delta_{m-1})(r_m + 1)]$ . It is not difficult to see that  $\pi$  is a permutation, and that  $(\sigma_i + \delta_i) < (\sigma_j + \delta_j)$  if and only if  $\sigma_i < \sigma_j$ , for  $1 \leq i \leq m - 1$ . Thus the conversion vector of  $\pi$  is  $r$ .  $\square$

**Example 2.7.** Consider the permutation  $[42351]$ . We can directly observe that the conversion vector is  $(0, 0, 1, 3, 0)$ . Conversely, since the proof in Proposition 2.6 is constructive, we can recursively recover the permutation as follows:

$$\begin{aligned}
[(0+1)] &= [1] \\
[(1+1)(0+1)] &= [21] \\
[(2+1)(1+0)(1+1)] &= [312] \\
[(3+0)(1+0)(2+0)(3+1)] &= [3124] \\
[(3+1)(1+1)(2+1)(4+1)(0+1)] &= [42351].
\end{aligned}$$

If the elements of the Apéry set of a numerical set  $I$  are known, the position vector can be determined without considering the enumeration of all the elements of  $I$ . To see this, let  $\text{Ap}(I, n) = \{\lambda_{x_0}, \lambda_{x_1}, \dots, \lambda_{x_{n-1}}\}$ . We write  $\lambda_{x_i} = nk_i + \pi_i$ , where  $0 \leq \pi_i < n$ . Notice that the elements of  $\text{Ap}(I, n)$  form a complete residue system modulo  $n$  and  $\pi_0 = 0$ . Thus,  $\pi = [\pi_1 \cdots \pi_{n-1}]$  is a permutation of the set  $\{1, \dots, n-1\}$  with a corresponding conversion vector  $r = (r_1, r_2, \dots, r_{n-1})$ .

**Theorem 2.8.** *Let  $I \in \Gamma_n$  and  $\text{Ap}(I, n) = \{w_0, w_1, \dots, w_{n-1}\}$ , where  $w_i < w_j$  whenever  $i < j$ . We write  $w_i = nk_i + \pi_i$ , with  $0 \leq \pi_i < n$ . Let  $r = (r_1, \dots, r_{n-1})$  be the conversion vector of  $\pi = [\pi_1 \pi_2 \cdots \pi_{n-1}]$ . Then  $\text{pv}_n(I) = (v_1, v_2, \dots, v_{n-1})$ , where  $v_1 = k_1 + 1$  and  $v_i = i(k_i - k_{i-1}) + (r_i - r_{i-1})$ , for  $2 \leq i \leq n-1$ .*

*Proof.* Let  $I = \{\lambda_0, \lambda_1, \dots\}$  and  $w_i = \lambda_{x_i}$ . Since  $0 = w_0 = \lambda_{x_0}$ , we have  $x_0 = 0$ . Next we show that  $x_i = ik_i - \sum_{j=0}^{i-1} k_j + r_i + 1$ , for  $1 \leq i \leq n-1$ . To do this, we compute the number of elements in  $I$  that are strictly less than  $nk_i$  for  $1 \leq i \leq n-1$ . The sequence  $(k_0, k_1, \dots, k_{n-1})$  is non-decreasing, and we set  $l$  to be the largest index such that  $k_i = k_l$ . Now  $s \in I$  and  $nk_l \leq s \leq nk_l + (n-1)$  if and only if  $s = \lambda_{x_j} + n(k_l - k_j)$  for some  $0 \leq j \leq l$ . Thus

$$\begin{aligned}
|\{0, 1, 2, \dots, nk_i - 1\} \cap I| &= |\{0, 1, 2, \dots, nk_l - 1\} \cap I| \\
&= \sum_{j=0}^l (k_l - k_j) \\
&= \sum_{j=0}^{i-1} (k_i - k_j) \\
&= ik_i - \sum_{j=0}^{i-1} k_j.
\end{aligned}$$

Next,  $s \in I$  and  $nk_i \leq s < \lambda_{x_i}$  if and only if  $s = \lambda_{x_j} + n(k_i - k_j) = nk_i + \pi_j$ , for some  $0 \leq j < i$  and  $\pi_j < \pi_i$ . There are  $r_i + 1$  such elements. Therefore, we have

$$\begin{aligned} x_i &= |\{\lambda_0, \lambda_1, \dots, \lambda_{x_i-1}\}| \\ &= ik_i - \sum_{j=0}^{i-1} k_j + r_i + 1 \end{aligned}$$

Recall that, by definition,  $v_i = x_i - x_{i-1}$ . Thus, the result now follows.  $\square$

Now we show that  $f : \Gamma_n \mapsto \mathbb{N}^{n-1}$  such that  $f(I) = \text{pv}_n(I)$  is a one-to-one correspondence by constructing the inverse map.

**Setup 2.9.** Let  $v = (v_1, v_2, \dots, v_{n-1}) \in \mathbb{N}^{n-1}$ . We will define two recursive sequences as follows:

- (1)  $t_1 = 0$  and  $t_i = (v_i + t_{i-1}) \bmod i$ , for  $2 \leq i \leq n-1$ .
- (2)  $l_1 = v_1 - 1$  and  $l_i = l_{i-1} + \frac{v_i + t_{i-1} - t_i}{i}$ , for  $2 \leq i \leq n-1$ .

Notice that  $0 \leq t_i \leq i-1$  for  $1 \leq i \leq n-1$ . Thus,  $(t_1, t_2, \dots, t_{n-1})$  is the conversion vector of a permutation  $\sigma = [\sigma_1 \sigma_2 \dots \sigma_{n-1}]$  according to Proposition 2.6. Now let

$$\mathcal{A}_v = \{0\} \cup \{nl_i + \sigma_i : 1 \leq i \leq n-1\}.$$

Since  $\sigma$  is a permutation on the set  $\{1, \dots, n-1\}$ ,  $\mathcal{A}_v$  is a complete residue system modulo  $n$  that contains 0. Thus,  $\mathcal{A}_v$  generates a numerical set  $I(\mathcal{A}_v) = \{w + kn : w \in \mathcal{A}_v, k \geq 0\}$  in  $\Gamma_n$ . We set  $g : \mathbb{N}^{n-1} \mapsto \Gamma_n$  such that  $g(v) = I(\mathcal{A}_v)$ .

**Theorem 2.10.** *The functions  $f : \Gamma_n \mapsto \mathbb{N}^{n-1}$  given by  $f(I) = \text{pv}_n(I)$  and  $g : \mathbb{N}^{n-1} \mapsto \Gamma_n$  by  $g(v) = I(\mathcal{A}_v)$  are inverse functions. Therefore, the function  $f$  is a one-to-one correspondence between numerical sets closed under addition by the element  $n$  and  $(n-1)$ -tuples of positive integers.*

*Proof.* Let  $I \in \Gamma_n$  with  $\text{Ap}(I, n) = \{0, w_1, \dots, w_{n-1}\}$ , and write  $w_i = nk_i + \pi_i$ , where  $0 \leq \pi_i < n$ . Also let  $v = \text{pv}_n(I)$  and  $\mathcal{A}_v = \{0\} \cup \{nl_i + \sigma_i : 0 \leq i \leq n-1\}$  as defined in Setup 2.9. It suffices to show that  $\pi_i = \sigma_i$  and  $k_i = l_i$  for all  $1 \leq i \leq n-1$ .

As noted before,  $[\pi_1 \pi_2 \dots \pi_{n-1}]$  is a permutation, and let  $r = (r_1, r_2, \dots, r_{n-1})$  be its conversion vector. By Theorem 2.8,  $r_1 = 0$ , and  $r_i = (v_i + r_{i-1}) - i(k_i - k_{i-1})$ . Since  $0 \leq r_i \leq i-1$ , it follows that  $r_i = (v_i + r_{i-1}) \bmod i$ , for  $1 \leq i \leq n-1$ . Referring to Setup 2.9, we find that  $r_i = t_i$ , for  $1 \leq i \leq n-1$ . Thus,  $r$  is the conversion vector of both  $[\pi_1 \pi_2 \dots \pi_{n-1}]$  and  $[\sigma_1 \sigma_2 \dots \sigma_{n-1}]$ , and we conclude the two permutations are equal.

Again, by Theorem 2.8 and Setup 2.9,  $k_1 = v_1 - 1 = l_1$ , and for  $2 \leq i \leq n-1$ ,

$$\begin{aligned} k_i - k_{i-1} &= \frac{v_i + r_{i-1} - r_i}{i} \\ &= \frac{v_i + t_{i-1} - t_i}{i} \\ &= l_i - l_{i-1} \end{aligned}$$

We conclude that  $k_i = l_i$ , for  $1 \leq i \leq n-1$ . This shows that  $g \circ f = \text{id}_{\Gamma_n}$ , and similarly, we obtain that  $f \circ g = \text{id}_{\mathbb{N}^{n-1}}$ . Therefore, the function  $f$  is a one-to-one correspondence.  $\square$

The next corollary is really a restatement of Theorem 2.10.

**Corollary 2.11.** *Every numerical set closed under addition by the element  $n \in \mathbb{N}$  has a unique position vector of length  $n - 1$ . In particular, no two numerical semigroups have the same position vector. Moreover, every vector of length  $n - 1$  with entries in  $\mathbb{N}$  is the position vector of a numerical set closed under addition by the element  $n$ .*

The next example demonstrates what we have developed in this section.

**Example 2.12.** Let  $S = \langle 6, 16, 20, 21, 29 \rangle$ . This is a numerical semigroup containing 6, and hence  $S \in \Gamma_6$ . We can compute

$$\text{Ap}(S, 6) = \{0, 16, 20, 21, 29, 37\} = \{0, 6(2) + 4, 6(3) + 2, 6(3) + 3, 6(4) + 5, 6(6) + 1\}.$$

The permutation [42351] has conversion vector  $(0, 0, 1, 3, 0)$ . Thus,

$$\begin{aligned} v_1 &= 2 + 1 = 3 \\ v_2 &= 2(3 - 2) + (0 - 0) = 2 \\ v_3 &= 3(3 - 3) + (1 - 0) = 1 \\ v_4 &= 4(4 - 3) + (3 - 1) = 6 \\ v_5 &= 5(6 - 4) + (0 - 3) = 7. \end{aligned}$$

So, we have  $\text{pv}_6(S) = (3, 2, 1, 6, 7)$ .

Conversely, suppose we start with  $v = (3, 2, 1, 6, 7) \in \mathbb{N}^5$ . According to Setup 2.9 and Theorem 2.10, the  $r_i$ 's are  $(0, 0, 1, 3, 0)$  and the  $k_i$ 's  $(2, 3, 3, 4, 6)$ . From the conversion vector  $(0, 0, 1, 3, 0)$ , we construct the corresponding permutation [42351] (see Example 2.7). Now,

$$\begin{aligned} w_0 &= 0 \\ w_1 &= 6(2) + 4 = 16 \\ w_2 &= 6(3) + 2 = 20 \\ w_3 &= 6(3) + 3 = 21 \\ w_4 &= 6(4) + 5 = 29 \\ w_5 &= 6(6) + 1 = 37. \end{aligned}$$

Thus,  $S = \{w_i + c_i n : c_i \geq 0\} = \langle 6, 16, 20, 21, 29 \rangle$ .

### 3. POSITION VECTORS OF NUMERICAL SEMIGROUPS

Now that we have a one-to-one correspondence between  $\mathbb{N}^{n-1}$  and numerical sets closed under addition by  $n$  established in the previous section, we want to know which position vectors correspond to semigroups. We provide a method for solving this problem and give an explicit answer for semigroups containing small numbers.

We begin with a necessary and sufficient condition for a complete residue system modulo  $n$  that contains 0 to be the Apéry set of a numerical semigroup. This result is similar to others contained in [2, 4], and the proof is omitted.

**Lemma 3.1.** *Let  $\mathcal{A} = \{w_0, w_1, \dots, w_{n-1}\}$  be a complete residue system modulo  $n$  that contains 0, where  $w_0 < w_1 < \dots < w_{n-1}$ . Then,  $I(\mathcal{A}) = \{w + kn : w \in \mathcal{A}, k \geq 0\}$  is a numerical semigroup if and only if  $w_i + w_j \geq w_l$  whenever  $w_i + w_j \equiv w_l \pmod{n}$  with  $0 < i \leq j < l$ .*

We can now translate Lemma 3.1 into a condition concerning the position vector, but first we need a few preliminary results.

**Lemma 3.2.** *Let  $I \in \Gamma_n$  be a numerical set with Apéry set  $\text{Ap}(I, n) = \{w_0, w_1, \dots, w_{n-1}\}$ , where  $w_0 < w_1 < \dots < w_{n-1}$ . We set  $w_i = nk_i + \pi_i$  for  $0 \leq i \leq n-1$ . If  $v = (v_1, \dots, v_{n-1})$  is the position vector of  $I$ , then  $k_1 = v_1 - 1$  and*

$$k_i - k_{i-1} = \left\lfloor \frac{v_i - 1}{i} \right\rfloor + \gamma_i,$$

where  $\gamma_1 = 0$  and

$$\gamma_i = \begin{cases} 0 & \text{if } \pi_{i-1} < \pi_i \\ 1 & \text{if } \pi_{i-1} > \pi_i \end{cases},$$

for  $2 \leq i \leq n-1$ .

*Proof.* It follows from Theorem 2.8 that  $k_1 = v_1 - 1$  and  $v_i = i(k_i - k_{i-1}) + (r_i - r_{i-1})$ , for  $2 \leq i \leq n-1$ , where  $r = (r_1, \dots, r_{n-1})$  is the conversion vector of  $\pi = [\pi_1 \dots \pi_{n-1}]$ . We rewrite this as  $v_i - 1 = i(k_i - k_{i-1} - \gamma_i) + (i\gamma_i + r_i - r_{i-1} - 1)$ . If we show that  $0 \leq i\gamma_i + r_i - r_{i-1} - 1 < i$  whenever  $2 \leq i \leq n-1$ , then it follows that

$$k_i - k_{i-1} = \left\lfloor \frac{v_i - 1}{i} \right\rfloor + \gamma_i.$$

First suppose that  $r_i > r_{i-1}$ . By the definition of the conversion vector, we have  $\pi_{i-1} < \pi_i$  and recall that  $0 \leq r_j \leq j-1$  for all  $1 \leq j \leq n-1$ . Thus,  $\gamma_i = 0$  and  $0 \leq r_i - r_{i-1} - 1 \leq i-2$ . Next, suppose that  $r_i \leq r_{i-1}$  so that  $\pi_{i-1} > \pi_i$ . Then  $\gamma_i = 1$  and  $1 \leq i + r_i - r_{i-1} - 1 \leq i-1$ , which completes the proof.  $\square$

The next proposition will lead to a convenient equivalence relation on the elements of  $\mathbb{N}^{n-1}$ , i.e., the position vectors of numerical sets in  $\Gamma_n$ .

**Proposition 3.3.** *Let  $v = (v_1, \dots, v_m)$  and  $z = (z_1, \dots, z_m)$  be two position vectors with associated conversion vectors and permutations denoted by  $r$  and  $\pi$  for  $v$ , and  $s$  and  $\sigma$  for  $z$ . Then the following are equivalent:*

- (1)  $v_i \equiv z_i \pmod{i}$  for all  $1 \leq i \leq m$
- (2)  $r = s$
- (3)  $\pi = \sigma$ .

*Proof.* For (1) implies (2), by Setup 2.9, the conversion vector depends on the position vector modulo  $i$  for the  $i$ -th entry. The equivalence of (2) and (3) follows from Proposition 2.6. Lastly, (2) implies (1) since, according to Theorem 2.8,  $v_i \equiv r_i - r_{i-1} \pmod{i}$ .  $\square$

**Definition 3.4.** We say two elements  $v = (v_1, \dots, v_m)$  and  $z = (z_1, \dots, z_m)$  of  $\mathbb{N}^m$  are *congruent*, denoted by  $v \sim z$ , if  $v_i \equiv z_i \pmod{i}$  for all  $1 \leq i \leq m$ . In this case,  $v$  and  $z$  have the same associated permutation  $\pi$  and are said to be in the *permutation class* defined by  $\pi$ .

We can now present the main result of this section.

**Theorem 3.5.** *Let  $(v_1, v_2, \dots, v_m)$  be a vector of positive integers in the permutation class defined by  $[\pi_1 \pi_2 \dots \pi_m]$ . Also set  $\gamma_i$  as in Lemma 3.2 and*

$$u_i = \left\lfloor \frac{v_i - 1}{i} \right\rfloor.$$

*Then  $(v_1, v_2, \dots, v_m)$  is the position vector of a numerical semigroup if and only if*

$$\sum_{x=1}^i (u_x + \gamma_x) + \frac{\pi_i + \pi_j - \pi_l}{m+1} \geq \sum_{x=j+1}^l (u_x + \gamma_x),$$

*whenever  $0 < i \leq j < l$  and  $\pi_i + \pi_j \equiv \pi_l \pmod{m+1}$ .*

*Proof.* Let  $(v_1, v_2, \dots, v_m)$  be the position vector of a numerical semigroup  $S$  with Apéry set  $\text{Ap}(S, m+1) = \{w_0, \dots, w_m\}$ , where  $w_i = (m+1)k_i + \pi_i$  for  $1 \leq i \leq m$ . If  $0 < i \leq j < l$  and  $\pi_i + \pi_j \equiv \pi_l \pmod{m+1}$ , then  $w_i + w_j \equiv w_l \pmod{m+1}$  and by Lemma 3.1, we have  $w_i + w_j \geq w_l$ . Thus,

$$\begin{aligned} w_i + w_j &\geq w_l \\ k_i + \frac{\pi_i + \pi_j - \pi_l}{m+1} &\geq k_l - k_j \\ \sum_{x=1}^i (k_x - k_{x-1}) + \frac{\pi_i + \pi_j - \pi_l}{m+1} &\geq \sum_{x=j+1}^l (k_x - k_{x-1}) \\ \sum_{x=1}^i (u_x + \gamma_x) + \frac{\pi_i + \pi_j - \pi_l}{m+1} &\geq \sum_{x=j+1}^l (u_x + \gamma_x). \end{aligned}$$

Essentially reversing these steps provides the converse argument and finishes the proof.  $\square$

The next example applies Theorem 3.5 to numerical semigroups containing 3.

**Example 3.6.** Every numerical set closed under addition by 3 belongs to one of two permutation classes, namely one defined by permutation [12] or the permutation [21]. We consider these two cases separately:

- (1) For the permutation [12], we have  $\pi_1 + \pi_1 \equiv \pi_2 \pmod{3}$ ,  $\gamma_1 = 0$ , and  $\gamma_2 = 0$ . Thus,

$$\begin{aligned} u_1 + \frac{1+1-2}{3} &\geq u_2 \\ u_1 &\geq u_2. \end{aligned}$$



(2) For the permutation [21], we have  $\pi_1 + \pi_1 \equiv \pi_2 \pmod{3}$ ,  $\gamma_1 = 0$ , and  $\gamma_2 = 1$ . Thus,

$$\begin{aligned} u_1 + \frac{2+2-1}{3} &\geq u_2 + 1 \\ u_1 &\geq u_2. \end{aligned}$$

We conclude that the vector  $(v_1, v_2)$  corresponds to a numerical semigroup if and only if  $u_1 \geq u_2$ , or equivalently,

$$v_1 - 1 \geq \left\lfloor \frac{v_2 - 1}{2} \right\rfloor.$$

Using the method derived from Theorem 3.5 and demonstrated in Example 3.6, we summarize the computational results for semigroups containing  $n$  where  $2 \leq n \leq 5$ . We omit the details.

**Theorem 3.7.** *Let  $(v_1, v_2, \dots, v_{n-1})$  be an  $(n-1)$ -tuple of positive integers and set*

$$u_i = \left\lfloor \frac{v_i - 1}{i} \right\rfloor.$$

*The following is a list necessary and sufficient conditions for  $v$  to be the position vector of the a numerical semigroup containing  $n$ , for  $2 \leq n \leq 5$ .*

- $n=2$ :  $(v_1)$  with no restriction
- $n=3$ :  $(v_1, v_2)$  such that  $u_1 \geq u_2$ .
- $n=4$ :  $(v_1, v_2, v_3)$  with restrictions given in Table 1

$\sim$ to one of	satisfying
$(1, 1, 1), (1, 2, 3)$	$u_1 \geq u_2$ and $u_1 \geq u_3$
$(1, 1, 2), (1, 2, 2)$	$u_1 \geq u_3$
$(1, 2, 1)$	$u_1 \geq u_2 + u_3$
$(1, 1, 3)$	$u_1 \geq u_2 + u_3 + 1$

TABLE 1. Restrictions for a 3-tuple to represent a semigroup

- $n=5$ :  $(v_1, v_2, v_3, v_4)$  with restrictions given in Table 2

In Theorem 3.7,  $n$  is an element of the semigroup. By adding an extra restriction, we can force  $n$  to be the multiplicity.

**Proposition 3.8.** *Let  $S$  be a semigroup containing  $n$  with position vector  $(v_1, v_2, \dots, v_{n-1})$ . Then  $n$  is the multiplicity of  $S$  if and only if  $v_1 > 1$ .*

*Proof.* We always have  $v_1 \geq 1$ . If  $v_1 > 1$ , then the first nonzero element of  $S$  is not in  $\text{Ap}(S, n)$ . Thus the first nonzero element of  $S$  cannot be smaller than  $n$ , and so  $n$  is the multiplicity of  $S$ . If  $v_1 = 1$ , then the first nonzero element of  $S$  is in  $\text{Ap}(S, n)$ . Thus the first nonzero element of  $S$  is smaller than  $n$ , and  $n$  is not the multiplicity of  $S$ .  $\square$

**Remark 3.9.** It follows that in Theorem 3.7, we can add the restriction  $u_1 > 0$  to ensure that  $n$  is the multiplicity of  $S$ .

$\sim$ to one of	satisfying
$(1, 1, 1, 1), (1, 1, 2, 2),$ $(1, 2, 2, 3), (1, 2, 3, 4)$	$u_1 \geq u_2, u_1 \geq u_3, u_1 \geq u_4,$ $and\ u_1 + u_2 \geq u_3 + u_4$
$(1, 2, 1, 2), (1, 2, 3, 1)$	$u_1 \geq u_2\ and\ u_1 \geq u_3 + u_4$
$(1, 1, 3, 3), (1, 1, 1, 4)$	$u_1 \geq u_2\ and\ u_1 \geq u_3 + u_4 + 1$
$(1, 1, 1, 2), (1, 2, 1, 4)$	$u_1 \geq u_2 + u_3, u_1 \geq u_4,$ $and\ u_1 + u_2 \geq u_3 + u_4$
$(1, 2, 3, 3), (1, 1, 3, 1)$	$u_1 \geq u_2 + u_3 + 1, u_1 \geq u_4,$ $and\ u_1 + u_2 \geq u_3 + u_4$
$(1, 1, 1, 3), (1, 2, 3, 2),$ $(1, 2, 2, 1), (1, 1, 2, 4),$ $(1, 1, 2, 3), (1, 2, 2, 2)$	$u_1 \geq u_2 + u_3 + u_4 + 1$
$(1, 1, 3, 4)$	$u_1 \geq u_2 + u_3 + u_4 + 2$
$(1, 2, 1, 1)$	$u_1 \geq u_2 + u_3 + u_4$
$(1, 1, 2, 1), (1, 2, 1, 3)$	$u_1 \geq u_2 + u_3\ and\ u_1 \geq u_3 + u_4$
$(1, 2, 2, 4), (1, 1, 3, 2)$	$u_1 \geq u_2 + u_3 + 1\ and\ u_1 \geq u_3 + u_4 + 1$

TABLE 2. Restrictions for a 4-tuple to represent a semigroup

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